

SPM BULLETIN

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1. EDITOR'S NOTE

We are now after the *First European Set Theory Meeting*, a historically important and well organized event. The talks gave the right blend of theory and applications of set theory. Thanks to Benedikt Loewe, Grzegorz Plebanek, Jouko Väänänen, and Boban Velickovic for the organization.

Following Jana Flašková's talk at this meeting, we have invited her to contribute a section to this issue. We thank her for her interesting contribution in Section 2 and in the *Problem of the Issue* section.

The list of problems at the end of the bulletin became longer than one page. We therefore removed the first few, and will continue this way unless some more problems are solved and their space becomes available...

A much better version of Shelah's paper showing that $\mathfrak{g} \leq \mathfrak{b}^+$ is now available at arxiv.org/abs/math/0612353

Enjoy,

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2. INVITED CONTRIBUTION: ULTRAFILTERS AND SMALL SETS

There have been several attempts to connect ultrafilters with families of “small” sets. Two of them — 0-points and \mathcal{I} -ultrafilters — were important for my Ph.D. thesis. The first one is due to Gryzlov [8]: an ultrafilter $\mathcal{U} \in \mathbb{N}^*$ is called a 0-point if for every one-to-one function $f : \mathbb{N} \rightarrow \mathbb{N}$ there exists a set $U \in \mathcal{U}$ such that $f[U]$ has asymptotic density zero. The second term was introduced by Baumgartner [1]: Let \mathcal{I} be a family of subsets of a set X such that \mathcal{I} contains all singletons and is closed under subsets. Given a free ultrafilter \mathcal{U} on \mathbb{N} , we say that \mathcal{U} is an \mathcal{I} -ultrafilter if for any $F : \mathbb{N} \rightarrow X$ there is $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

In my Ph.D. thesis [3] (on which all my papers are more less based) I studied \mathcal{I} -ultrafilters in the setting $X = \mathbb{N}$ and \mathcal{I} is an ideal on \mathbb{N} or another family of “small” subsets of \mathbb{N} that contains finite sets and is closed under subsets. As \mathcal{I} were considered the ideal of sets with asymptotic density zero $\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n \rightarrow \infty} \frac{|A \cap n|}{n} = 0\}$, the summable ideal $\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$ or the family of (almost) thin sets and (SC)-sets.

Here are the (probably not common) definitions: We say that $A \subseteq \mathbb{N}$ with an increasing enumeration $A = \{a_n : n \in \mathbb{N}\}$ is

thin: if $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = 0$;

almost thin: if $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} < 1$;

(SC)-set: if $\lim_{n \rightarrow \infty} a_{n+1} - a_n = \infty$.

In the thesis various examples of \mathcal{I} -ultrafilters for all these (and also some other) families \mathcal{I} are constructed under additional set theoretic assumptions.

In my first paper [4] it is shown that thin sets and almost thin sets actually determine the same class of \mathcal{I} -ultrafilters and there is a proof that the existence of these ultrafilters is independent of ZFC. The relation between this class of ultrafilters and selective ultrafilters or Q -points is studied. Some construction made in the paper under CH were proved in the thesis assuming MA_{ctble} .

The next paper [5] focuses on \mathcal{I} -ultrafilters where \mathcal{I} is the summable ideal $\mathcal{I}_{1/n}$ or the density ideal \mathcal{Z}_0 . The relation between these two classes of ultrafilters is shown and also the relation to the class of P -points. Assuming CH or MA_{ctble} several examples of these ultrafilters are constructed. Again, stronger versions of some of the results can be found in the thesis.

One of the few ZFC results in my thesis is the following: There exists an ultrafilter $\mathcal{U} \in \mathbb{N}^*$ such that for every one-to-one function $f : \mathbb{N} \rightarrow \mathbb{N}$ there exists a set

$U \in \mathcal{U}$ with $f[U]$ in the summable ideal. This theorem strengthens Gryzlov's result concerning the existence of 0-points and it was published also separately in [6].

Connections between various \mathcal{I} -ultrafilters and some well-known ultrafilters such as P -points were studied in two sections of my thesis. It is known that P -points can be described as \mathcal{I} -ultrafilters in two different ways: If $X = 2^{\mathbb{N}}$ then P -points are precisely the \mathcal{I} -ultrafilters for \mathcal{I} consisting of all finite and converging sequences, if $X = \omega_1$ then P -points are precisely the \mathcal{I} -ultrafilters for $\mathcal{I} = \{A \subseteq \omega_1 : A \text{ has order type } \leq \omega\}$. My latest paper [7] deals with the question whether there is a family \mathcal{I} of subsets of natural numbers such that P -points are precisely the \mathcal{I} -ultrafilters. However, only some partial answers are given.

During the 1st European Set Theory Meeting in Będlewo I gave a talk "On sums and products of certain \mathcal{I} -ultrafilters". As the title suggests it was a summary of my knowledge about sums and products of some \mathcal{I} -ultrafilters. The slides and notes with proofs on which the talk was based (as well as my Ph.D. thesis) are available online on my webpage:

<http://home.zcu.cz/~flaskova>

Jana Flašková

Department of Mathematics, University of West Bohemia

3. RESEARCH ANNOUNCEMENTS

3.1. Inverse Systems and I-Favorable Spaces. Let X be a compact space. Player I has a winning strategy in the open-open game played on X if, and only if X can be represented as a limit of σ -complete inverse system of compact metrizable spaces with skeletal bonding maps.

<http://arxiv.org/abs/0706.3815>

Andrzej Kucharski and Szymon Plewik

3.2. Combinatorial and hybrid principles for σ -directed families of countable sets modulo finite. We consider strong combinatorial principles for σ -directed families of countable sets in the ordering by inclusion modulo finite, e.g. P -ideals of countable sets. We try for principles as strong as possible while remaining compatible with CH, and we also consider principles compatible with the existence of nonspecial Aronszajn trees. The main thrust is towards abstract principles with game theoretic formulations. Some of these principles are purely combinatorial, while the ultimate principles are primarily combinatorial but also have aspects of forcing axioms.

<http://arxiv.org/abs/0706.3729>

James Hirschorn

3.3. A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension. We study the extension of the Kechris-Solecki-Todorćević dichotomy on analytic graphs to dimensions higher than 2. We prove that the extension is possible in any dimension, finite or infinite. The original

proof works in the case of the finite dimension. We first prove that the natural extension does not work in the case of the infinite dimension, for the notion of continuous homomorphism used in the original theorem. Then we solve the problem in the case of the infinite dimension. Finally, we prove that the natural extension works in the case of the infinite dimension, but for the notion of Baire-measurable homomorphism.

<http://arxiv.org/abs/0707.1313>

Dominique Lecomte

3.4. A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension. We study the extension of the Kechris-Solecki-Todorćević dichotomy on analytic graphs to dimensions higher than 2. We prove that the extension is possible in any dimension, finite or infinite. The original proof works in the case of the finite dimension. We first prove that the natural extension does not work in the case of the infinite dimension, for the notion of continuous homomorphism used in the original theorem. Then we solve the problem in the case of the infinite dimension. Finally, we prove that the natural extension works in the case of the infinite dimension, but for the notion of Baire-measurable homomorphism.

<http://arxiv.org/abs/0707.1313>

Dominique Lecomte

3.5. Large continuum, oracles. Our main theorem is about iterated forcing for making the continuum larger than \aleph_2 . We present a generalization of [math.LO/0303294](#) which is dealing with oracles for random, etc., replacing \aleph_1, \aleph_2 by λ, λ^+ (starting with $\lambda = \lambda^{<\lambda} > \aleph_1$). Well, instead of properness we demand absolute c.c.c. So we get, e.g. the continuum is λ^+ but we can get $\text{cov}(\mathcal{M}) = \lambda$. We give some applications. As in [math.LO/0303294](#), it is a “partial” countable support iteration but it is c.c.c.

<http://arxiv.org/abs/0707.1818>

Saharon Shelah

3.6. Borel hierarchies in infinite products of Polish spaces. Let H be a product of countably infinite number of copies of an uncountable Polish space X . Let Σ_ξ ($\bar{\Sigma}_\xi$) be the class of Borel sets of additive class ξ for the product of copies of the discrete topology on X (the Polish topology on X), and let $\mathcal{B} = \bigcup_{\xi < \omega_1} \bar{\Sigma}_\xi$. We prove in the Lévy–Solovay model that $\bar{\Sigma}_\xi = \Sigma_\xi \cap \mathcal{B}$ for $1 \leq \xi < \omega_1$.

<http://arxiv.org/abs/0707.1967>

Rana Barua and Ashok Maitra

3.7. A game for the Borel functions. Abstract. We present an infinite game that characterizes the Borel functions on Baire Space.

www.illc.uva.nl/Publications/ResearchReports/PP-2006-24.text.pdf

Brian Semmes

3.8. On some problems in general topology. We prove that Arhangel'skii's problem has a consistent positive answer: If V is a model of CH, then for some \aleph_1 -complete \aleph_2 -c.c. forcing notion P of cardinality \aleph_2 , we have that P forces “CH and there is a Lindelöf regular topological space of size \aleph_2 with clopen basis with every point of pseudo-character \aleph_0 (i.e., each singleton is the intersection of countably many open sets)”.

Also, we prove the consistency of: $\text{CH} + 2^{\aleph_1} > \aleph_2$ + “there is no space as above with \aleph_2 points” (starting with a weakly compact cardinal).

Appeared in: *Set Theory*, Boise ID, 1992–1994, Contemporary Mathematics, vol. 192, 91–101.

<http://arxiv.org/abs/0708.1981>
Saharon Shelah

4. PROBLEM OF THE ISSUE

Definitions not stated below can be found in Section 2 above. Some more definitions concerning ultrafilters can be found in [2].

The problem of this issue concerns products of \mathcal{I} -ultrafilters for the case $X = \mathbb{N}$ and \mathcal{I} is an ideal on \mathbb{N} .

Definition 4.1. If \mathcal{U} and \mathcal{V} are ultrafilters on \mathbb{N} then $\mathcal{U} \cdot \mathcal{V}$ is the ultrafilter on $\mathbb{N} \times \mathbb{N}$ defined by $M \in \mathcal{U} \cdot \mathcal{V}$ if and only if $\{n : \{m : \langle n, m \rangle \in M\} \in \mathcal{V}\} \in \mathcal{U}$. Since isomorphic ultrafilters can be identified we may regard $\mathcal{U} \cdot \mathcal{V}$ as an ultrafilter on \mathbb{N} . The ultrafilter $\mathcal{U} \cdot \mathcal{V}$ is called the *product of ultrafilters \mathcal{U} and \mathcal{V}* .

Definition 4.2. Let \mathcal{I} be an ideal on \mathbb{N} . We say that \mathcal{I} -ultrafilters are *closed under products* if the product of two arbitrary \mathcal{I} -ultrafilters is again an \mathcal{I} -ultrafilter.

For a P -ideal \mathcal{I} the class of \mathcal{I} -ultrafilters is closed under products [3]. However, not much is known for other ideals. For example, if \mathcal{I} is the ideal generated by thin sets or the ideal generated by (SC) -sets then \mathcal{I} -ultrafilters are not closed under products [3]. In fact, even more is true.

Theorem 4.3 ([3]). *For every $\mathcal{U} \in \mathbb{N}^*$ the ultrafilter $\mathcal{U} \cdot \mathcal{U}$ is not an (SC) -ultrafilter (thin ultrafilter).*

This property shares the class of all P -points (the partition $\{\{n\} \times \mathbb{N} : n \in \mathbb{N}\}$ witnesses the fact that no product of two free ultrafilters is a P -point), but it is consistent with ZFC that there exist thin ultrafilters (and hence (SC) -ultrafilters) which are not P -points [3].

Another example of an ideal which is not a P -ideal is the following.

Definition 4.4. The *van der Waerden ideal* \mathcal{W} is the family of all $A \subseteq \mathbb{N}$ such that A does *not* contain arithmetic progressions of arbitrary length.

Problem 4.5. *Are \mathcal{W} -ultrafilters closed under products?*

A positive answer would provide (consistent) examples of \mathcal{W} -ultrafilters that are neither P -points nor (SC)-ultrafilters (and thin ultrafilters). But I expect rather a negative answer.

Jana Flašková

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- [7] J. Flašková, *A note on \mathcal{I} -ultrafilters and P -points*, submitted to *Proceedings of the Winter School in Abstract Analysis 2007*.
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5. UNSOLVED PROBLEMS FROM EARLIER ISSUES

- Issue 4.** Does $S_1(\Omega, T)$ imply $U_{fin}(\Gamma, \Gamma)$?
- Issue 5.** Is $\mathfrak{p} = \mathfrak{p}^*$? (See the definition of \mathfrak{p}^* in that issue.)
- Issue 6.** Does there exist (in ZFC) an uncountable set satisfying $S_1(\mathcal{B}_\Gamma, \mathcal{B})$?
- Issue 8.** Does $X \notin \text{NON}(\mathcal{M})$ and $Y \notin \text{D}$ imply that $X \cup Y \notin \text{COF}(\mathcal{M})$?
- Issue 9 (CH).** Is $\text{Split}(\Lambda, \Lambda)$ preserved under finite unions?
- Issue 10.** Is $\text{cov}(\mathcal{M}) = \mathfrak{od}$? (See the definition of \mathfrak{od} in that issue.)
- Issue 11.** Does $S_1(\Gamma, \Gamma)$ always contain an element of cardinality \mathfrak{b} ?
- Issue 12.** Could there be a Baire metric space M of weight \aleph_1 and a partition \mathcal{U} of M into \aleph_1 meager sets where for each $\mathcal{U}' \subset \mathcal{U}$, $\bigcup \mathcal{U}'$ has the Baire property in M ?
- Issue 14.** Does there exist (in ZFC) a set of reals X of cardinality \mathfrak{d} such that all finite powers of X have Menger's property $S_{fin}(\mathcal{O}, \mathcal{O})$?
- Issue 15.** Can a Borel non- σ -compact group be generated by a Hurewicz subspace?
- Issue 16 (MA).** Is there an uncountable $X \subseteq \mathbb{R}$ satisfying $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$?
- Issue 17 (CH).** Is there a totally imperfect X satisfying $U_{fin}(\mathcal{O}, \Gamma)$ that can be mapped continuously onto $\{0, 1\}^\mathbb{N}$?
- Issue 18 (CH).** Is there a Hurewicz X such that X^2 is Menger but not Hurewicz?
- Issue 19.** Does the Pytkeev property of $C_p(X)$ imply the Menger property of X ?
- Issue 20.** Does every hereditarily Hurewicz space satisfy $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$?
- Issue 21 (CH).** Is there a Rothberger-bounded $G \leq \mathbb{Z}^\mathbb{N}$ such that G^2 is not Menger-bounded?
- Issue 22.** Let \mathcal{W} be the van der Waerden ideal. Are \mathcal{W} -ultrafilters closed under products?

Previous issues. The previous issues of this bulletin, which contain general information (first issue), basic definitions, research announcements, and open problems (all issues) are available online, at <http://front.math.ucdavis.edu/search?&t=%22SPM+Bulletin%22>

Contributions. Please submit your contributions (announcements, discussions, and open problems) by e-mailing us. It is preferred to write them in L^AT_EX. The authors are urged to use as standard notation as possible, or otherwise give the definitions or a reference to where the notation is explained. Contributions to this bulletin would not require any transfer of copyright, and material presented here can be published elsewhere.

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